

Fundamental Formulas of  
**Static Electric and Magnetic Fields**

(Derived from *Field and Wave Electromagnetics* by *David K. Cheng*)

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# 1 Vector analysis

## 1-1 Orthogonal coordinate systems

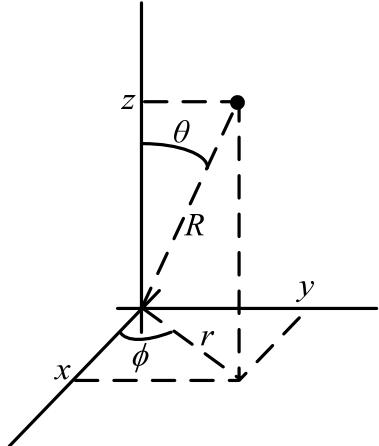


Fig. 1-1 Coordinates

$$\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2 = \hat{\mathbf{u}}_3 \quad (1.1)$$

### 1-1-1 Rectangular coordinates

$$(u_1, u_2, u_3) = (x, y, z) \quad (1.2)$$

$$h_1 = h_2 = h_3 = 1 \quad (1.3)$$

### 1-1-2 Cylindrical coordinates

$$(u_1, u_2, u_3) = (r, \phi, z) \quad (1.4)$$

$$h_1 = 1$$

$$h_2 = r \quad (1.5)$$

$$h_3 = 1$$

### 1-1-3 Spherical coordinates

$$(u_1, u_2, u_3) = (R, \theta, \phi) \quad (1.6)$$

$$h_1 = 1$$

$$h_2 = R \quad (1.7)$$

$$h_3 = R \sin \theta$$

## 1-2 Differential operators

### 1-2-1 Gradient

$$\nabla V = \hat{\mathbf{u}}_1 \frac{1}{h_1} \frac{\partial V}{\partial u_1} + \hat{\mathbf{u}}_2 \frac{1}{h_2} \frac{\partial V}{\partial u_2} + \hat{\mathbf{u}}_3 \frac{1}{h_3} \frac{\partial V}{\partial u_3} \quad (1.8)$$

Directional derivative

$$\frac{dV}{dl} = (\nabla V) \cdot \hat{\mathbf{l}} \quad (1.9)$$

$$dV = (\nabla V) \cdot d\mathbf{l} \quad (1.10)$$

### 1-2-2 Divergence

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left[ \frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right] \quad (1.11)$$

### 1-2-3 Curl

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{u}}_1 & h_2 \hat{\mathbf{u}}_2 & h_3 \hat{\mathbf{u}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \quad (1.12)$$

## 1-3 Theorems

### 1-3-1 Divergence theorem

$$\int_{V'} \nabla \cdot \mathbf{A} dv = \oint_S \mathbf{A} \cdot d\mathbf{s} \quad (1.13)$$

### 1-3-2 Stokes' theorem

$$\oint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \int_C \mathbf{A} \cdot d\mathbf{l} \quad (1.14)$$

## 1-4 Vector identities

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \quad (1.15)$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \quad (1.16)$$

$$\nabla \times (\nabla V) = 0 \quad (1.17)$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = \mathbf{0} \quad (1.18)$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f \quad (1.19)$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \quad (1.20)$$

$$\nabla \times (f\mathbf{A}) = f\nabla \times \mathbf{A} + (\nabla f) \times \mathbf{A} \quad (1.21)$$

$$\nabla^2 V \equiv \nabla \cdot \nabla V \quad (1.22)$$

$$\nabla^2 \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla \times \nabla \times \mathbf{A} \quad (1.23)$$

## 2 Static electric fields

### 2-1 Coulomb's law

$$\mathbf{F}_{12} = \hat{\mathbf{R}}_{12} \frac{1}{4\pi\epsilon} \frac{q_1 q_2}{R_{12}^2} \quad (2.1)$$

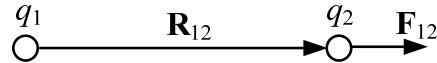


Fig. 2-1 Electric force between two charges

### 2-2 Electric field intensity

$$\mathbf{E} = \lim_{q \rightarrow 0} \frac{\mathbf{F}}{q} \quad (2.2)$$

$$\mathbf{F} = q\mathbf{E} \quad (2.3)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \quad (2.4)$$

$$\nabla \times \mathbf{E} = \mathbf{0} \quad (2.5)$$

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\epsilon} \int_{V'} \rho dV = \frac{Q}{\epsilon} \quad (2.6)$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad (2.7)$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon} \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|^3} \quad (2.8)$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \sum_{k=1}^N \frac{q_k (\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \quad (2.9)$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon} \int_{V'} \frac{(\mathbf{R} - \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|^3} \rho dV' \quad (2.10)$$

$$\mathbf{E} = -\nabla V \quad (2.11)$$

### 2-3 Electric flux density

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (2.12)$$

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} \quad (2.13)$$

$$\nabla \cdot \mathbf{D} = \rho \quad (2.14)$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{V'} \rho dV = Q \quad (2.15)$$

## 2-4 Polarization vector

$$\mathbf{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \quad (2.16)$$

$n$ : number of atoms per unit volume

$$\mathbf{p} = q\mathbf{d} \quad (2.17)$$

$$\rho_p = -\nabla \cdot \mathbf{P} \quad (2.18)$$

$$\rho_{ps} = \mathbf{P} \cdot \hat{\mathbf{n}} \quad (2.19)$$

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \epsilon_0 (\epsilon_r - 1) \mathbf{E} \quad (2.20)$$

## 2-5 Electric potential

$$V_2 - V_1 = - \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \quad (2.21)$$

$$V = \frac{q}{4\pi\epsilon R} \quad (2.22)$$

$$V = \frac{1}{4\pi\epsilon} \sum_{k=1}^N \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \quad (2.23)$$

$$V = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho dv'}{|\mathbf{R} - \mathbf{R}'|} \quad (2.24)$$

$$V = \frac{W}{q} \quad (2.25)$$

## 2-6 Electric dipole

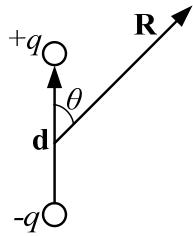


Fig. 2-2 Electric dipole

$$\mathbf{p} = q\mathbf{d} \quad (2.26)$$

$$\mathbf{E} \equiv \frac{q}{4\pi\epsilon R^3} \left[ 3 \frac{\mathbf{R} \cdot \mathbf{d}}{R^2} \mathbf{R} - \mathbf{d} \right] = \frac{1}{4\pi\epsilon R^3} \left[ 3 \frac{\mathbf{R} \cdot \mathbf{p}}{R^2} \mathbf{R} - \mathbf{p} \right] \quad R \gg d \quad (2.27)$$

$$\mathbf{E} \equiv \frac{p}{4\pi\epsilon R^3} \left[ \hat{\mathbf{R}} 2 \cos \theta + \hat{\mathbf{\theta}} \sin \theta \right] \quad R \gg d \quad (2.28)$$

$$V = \frac{\mathbf{p} \cdot \hat{\mathbf{R}}}{4\pi\epsilon R^2} \quad (2.29)$$

### 3 Electric currents

#### 3-1 Electric current density

$$\mathbf{J} = \sum_{i=e,h} n_i q_i \mathbf{u}_i \quad (3.1)$$

$n$ : number of charge carriers per unit volume

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \quad (3.2)$$

$$\mathbf{J} = \sigma \mathbf{E} \quad (3.3)$$

#### 3-2 Resistance of a straight piece

$$R = \frac{V}{I} = \frac{l}{\sigma S} \quad (3.4)$$

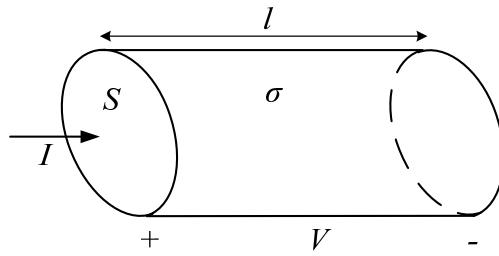


Fig. 3-1 Homogeneous conductor with a constant cross-section

#### 3-3 Equation of continuity

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (3.5)$$

#### 3-4 Power dissipation

$$P = \int_{V'} \mathbf{E} \cdot \mathbf{J} dV \quad (3.6)$$

#### 3-5 Governing equations at steady state

$$\nabla \cdot \mathbf{J} = 0 \quad (3.7)$$

$$\nabla \times \left( \frac{\mathbf{J}}{\sigma} \right) = \mathbf{0} \quad \text{where no nonconservative energy source exists} \quad (3.8)$$

### 3-6 Boundary conditions for steady currents

$$\hat{\mathbf{n}} \cdot (\mathbf{J}_1 - \mathbf{J}_2) = 0 \quad (3.9)$$

$$\hat{\mathbf{n}} \times \left( \frac{\mathbf{J}_1}{\sigma_1} - \frac{\mathbf{J}_2}{\sigma_2} \right) = \mathbf{0} \quad (3.10)$$

## 4 Static magnetic fields

### 4-1 Lorentz's force equation

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (4.1)$$

### 4-2 Magnetic flux density

#### 4-2-1 Governing equations in static cases

$$\nabla \cdot \mathbf{B} = 0 \quad (4.2)$$

$$\nabla \times \mathbf{B} = \mu \mathbf{J} \quad (4.3)$$

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \quad (4.4)$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu I \quad (4.5)$$

#### 4-2-2 Biot-Savart's law

$$\mathbf{B} = \frac{\mu I}{4\pi} \oint_C \frac{dl' \times (\mathbf{R} - \mathbf{R}')}{|\mathbf{R} - \mathbf{R}'|^3} \quad (4.6)$$

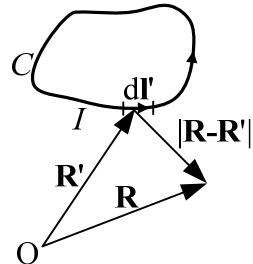


Fig. 4-1 A wire loop carrying current  $I$

#### 4-2-3 Other relations

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (4.7)$$

$$\mathbf{B} = \mu \mathbf{H} \quad (4.8)$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) \quad (4.9)$$

### 4-3 Magnetic field intensity

$$\mathbf{H} = \frac{\mathbf{B}}{\mu} \quad (4.10)$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \quad (4.11)$$

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (4.12)$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I \quad (4.13)$$

### 4-4 Vector magnetic potential

$$\nabla \times \mathbf{A} = \mathbf{B} \quad (4.14)$$

#### 4-4-1 Coulomb's gauge

$$\nabla \cdot \mathbf{A} = 0 \quad (4.15)$$

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} \quad (4.16)$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \quad (4.17)$$

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J} dv'}{|\mathbf{R} - \mathbf{R}'|} \quad (4.18)$$

$$\mathbf{A} = \frac{\mu I}{4\pi} \oint_{C'} \frac{d\mathbf{l}'}{|\mathbf{R} - \mathbf{R}'|} \quad (4.19)$$

### 4-5 Magnetic flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad (4.20)$$

$$\Phi = \int_C \mathbf{A} \cdot d\mathbf{l} \quad (4.21)$$

### 4-6 Magnetic dipole

$$\mathbf{m} = IS\hat{\mathbf{n}} \quad (4.22)$$

$$\mathbf{A} = \frac{\mu}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{R}}}{R^2} \quad (4.23)$$

$$\mathbf{B} = \frac{\mu m}{4\pi R^3} \left[ \hat{\mathbf{R}} 2 \cos \theta + \hat{\theta} \sin \theta \right] \quad (4.24)$$

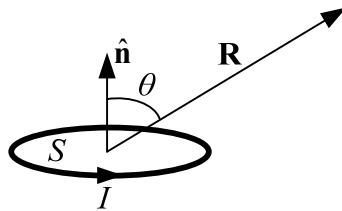


Fig. 4-2 Magnetic dipole

## 4-7 Magnetization vector

$$\mathbf{M} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{m}_k}{\Delta v} \quad (4.25)$$

$n$ : number of atoms per unit volume

$$\mathbf{M} = \chi_m \mathbf{H} \quad (4.26)$$

$$\mathbf{M} = \frac{\mathbf{B}}{\mu_0} - \mathbf{H} \quad (4.27)$$

### 4-7-1 Magnetization current densities

$$\mathbf{J}_m = \nabla \times \mathbf{M} \quad (4.28)$$

$$\mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{n}} \quad (4.29)$$

### 4-7-2 Magnetization charge densities

$$\rho_m = -\nabla \cdot \mathbf{M} \quad (4.30)$$

$$\rho_{ms} = \mathbf{M} \cdot \hat{\mathbf{n}} \quad (4.31)$$

## 4-8 Magnetic circuits

$$NI = \mathfrak{R}\Phi \quad (4.32)$$

### 4-8-1 Reluctance

$$\mathfrak{R} = \frac{l}{\mu S} \quad (4.33)$$

## 4-9 Boundary conditions for magneto-static fields

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0 \quad (4.34)$$

$$\hat{\mathbf{n}} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (4.35)$$

## 5 Useful integrals

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) \quad (5.1)$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left( \frac{x-a}{x+a} \right) \quad (5.2)$$

$$\int \frac{x dx}{x^2 \pm a^2} = \frac{1}{2} \ln(x^2 \pm a^2) \quad (5.3)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln \left( x + \sqrt{x^2 \pm a^2} \right) \quad (5.4)$$

$$\int \frac{x dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2} \quad (5.5)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left( \frac{x}{a} \right) \quad (5.6)$$

$$\int \frac{x dx}{\sqrt{a^2 - x^2}} = -\sqrt{a^2 - x^2} \quad (5.7)$$

$$\int \frac{dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = \pm \frac{x}{a^2 \sqrt{x^2 \pm a^2}} \quad (5.8)$$

$$\int \frac{x dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{x^2 \pm a^2}} \quad (5.9)$$

$$\int \frac{x^2 dx}{(x^2 \pm a^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{x^2 \pm a^2}} + \ln \left( x + \sqrt{x^2 \pm a^2} \right) \quad (5.10)$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x}{a} \right) \quad (5.11)$$

$$\int \ln \left( x + \sqrt{x^2 \pm a^2} \right) dx = -\sqrt{x^2 \pm a^2} + x \ln \left( x + \sqrt{x^2 \pm a^2} \right) \quad (5.12)$$

$$\begin{aligned} \int \ln \left( b + \sqrt{x^2 + a^2} \right) dx &= \\ &\sqrt{a^2 - b^2} \left[ \tan^{-1} \left( \frac{x}{\sqrt{a^2 - b^2}} \right) - \tan^{-1} \left( \frac{bx}{\sqrt{a^2 - b^2} \sqrt{x^2 + a^2}} \right) \right] \quad (5.13) \\ &+ x \left[ -1 + \ln \left( b + \sqrt{x^2 + a^2} \right) + b \ln \left( x + \sqrt{x^2 + a^2} \right) \right] \end{aligned}$$